SOLUTIONS TO SELECTED QUESTIONS IN HOMEWORK 8

MATH 241

19.2.4

Proof. $\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$, and $\frac{1}{(1-z)^3} = \frac{1}{2}(\frac{1}{1-z})''$, so differentiate term by term we get $\frac{1}{(1-z)^3} = \frac{1}{2}\sum_{k=2}^{\infty} k(k-1)z^{k-2}$. Therefore the Maclaurin series of $\frac{z}{(1-z)^3}$ is

$$z \cdot \frac{1}{2}\sum_{k=2}^{\infty} k(k-1)z^{k-2} = \frac{1}{2}\sum_{k=2}^{\infty} k(k-1)z^{k-1}$$

The only singularity is $z = 1$, the distance from the center 0 to the singularity is 1, so the radius of convergence of the Taylor series is 1. □

19.2.13

Proof. $f(z) = \frac{1}{z} = \frac{1}{1+(z-1)} = \frac{1}{1-(-(z-1))} = \sum_{k=0}^{\infty}(-(z-1))^k = \sum_{k=0}^{\infty}(-1)^k(z-1)^k$. The singularity is $z = 0$, so radius of convergence is $|1-0| = 1$. □

19.2.17

Proof.

$$f(z) = \frac{z-1}{3-z} = \frac{z-1}{2-(z-1)} = \frac{1}{2}\frac{z-1}{1-\frac{z-1}{2}} = \frac{1}{2}(z-1)\sum_{k=0}^{\infty}\frac{(z-1)^k}{2^k} = \sum_{k=0}^{\infty}\frac{(z-1)^{k+1}}{2^{k+1}}$$

Singularity is 3, center is at 1, so the radius of convergence is $|1-3| = 2$. □

19.2.27

Proof. Singularities are $z = \pm i$, center is $2+5i$, so the radius of convergence is $|(2+5i)-i| = |2+4i| = \sqrt{2^2+4^2} = 2\sqrt{5}$. □

Spring 10, #4

Proof. $\frac{1}{z-2} = \frac{1}{1+(z-3)} = \sum_{k=0}^{\infty}(-1)^k(z-3)^k$, so

$$\frac{1}{(z-2)^2} = -(\frac{1}{z-2})' = -\sum_{k=1}^{\infty}(-1)^k k(z-3)^{k-1} = 1 - 2(z-3) + 3(z-3)^2 - 4(z-1)^3 + 5(z-1)^4 - \cdots$$

On the other hand,

$$\frac{1}{z-4} = \frac{1}{-1+(z-3)} = -\frac{1}{1-(z-3)} = -\sum_{k=0}^{\infty}(z-3)^k = -1 - (z-3) - (z-3)^2 - (z-3)^3 - (z-3)^4 - \cdots$$

So their product's degree four term will be

$$1\cdot(-(z-3)^4)+(-2(z-3))\cdot(-(z-3)^3)+(3(z-3)^2)\cdot(-(z-3)^2)+(-4(z-1)^3)\cdot(-(z-3))+(5(z-1)^4)\cdot(-1) = -3(z-1)^4$$

□

Spring 08, extra credit question

Proof. Since $f(z)$ is analytic on the disk, the Taylor series converges to its value, so $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)(1)}}{k!}(z-1)^k$ on the disk $\{z||z-1| < 2\}$. But all the derivatives are zero, so $f(z) = 0$ on this disk. □